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Letter

Test of the relation between temporal and spatial Q by Knopoff et al.

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ABSTRACT

The quality factor is a dimensionless measure of the energy loss per cycle of wave modes in an attenuation medium. Accurate measurement is important in various fields, from seismological studies to detect zones of partial melting to the geophysics of reservoirs to study rock properties such as porosity, fluid properties and saturation and permeability. In seismology, the quality factors measured for normal (standing) modes and propagating waves differ, as well those of equivalent experiments based on resonant rods and ultrasonic pulses performed in the laboratory. These measurements result in temporal and spatial quality factors respectively. A relationship between these two different quality factors and between the corresponding attenuation factors was proposed by Knopoff et al. sixty years ago. The conversion factor is basically the ratio between the phase velocity and the group velocity, while for the attenuation factor is the group velocity. We test these relations, which hold for low-loss solids, for body waves, using a Kelvin–Voigt rheology and a constant Q model, which provide explicit expressions of the temporal and spatial quality factors and velocities involved in these relations. The proposed theory provides the basis for a complete characterization of temporal and spatial quality factors and velocity dispersion based on arbitrary stress–strain relationships.

1. Introduction

In anelastic media, there are standing (stationary) and propagating (traveling) waves associated with temporal and spatial amplitude decay. The distinction between temporal and spatial quality factor is important. In a resonant bar process (see [1], App. B), a sample is vibrated in one of its eigenmodes so that a standing wave is formed in the sample. Such standing waves result from the interference of propagating waves moving in opposite directions as a result of successive reflections at the ends of the sample. In contrast, the wave propagation method (e.g. [2–5]) uses a wave that moves through the specimen in only one direction at a time; interference effects then do not occur. The quality factors obtained with the two methods are different.

Knopoff et al. [6] showed that the temporal and spatial quality factors, Q_T and Q, respectively, are related as

$$Q_T = \frac{v_p}{v_g} Q, \tag{1}$$

where v_g and v_p are the spatial group and phase velocities. Moreover, they showed that

$$\omega_I = v_g \alpha, \tag{2}$$

where ω_I and α are the temporal and spatial attenuation factors, such that the standing and propagating waves attenuate as $\exp(-\omega_I t)$ and $\exp(-\alpha x)$, respectively, where t and x are the time and spatial variables, respectively. They assumed Q_T and $Q \gg 1$ as well as

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isotropy and 1D propagation in their calculation. Knopoff et al. [6] conclude that the Q_T's measured from observations of the Earth's free modes of vibration must be modified when compared with the Q's at the same periods obtained from experiments with propagating waves. This fact is important, and notable scientists such as Knopoff, Aki and Ben-Menahem have chosen to establish this relationship, which has gone relatively unnoticed in subsequent literature. In the following, we find that the group velocity performs better in the relationship (2).

In what follows, we illustrate the actual relation between the temporal and spatial quality factor using the Kelvin–Voigt stress–strain relation and constant-Q model. The first allows us to describe wavelike and diffusion-like behaviors in a finite frequency band ([0, 300] Hz) in the example. The Kelvin–Voigt rheology has been successfully used to model seismic waves at the Earth's surface [7]. A detailed description of its frequency- and time-domain functions can be found in [8], Section 2.4) and [2], Section 2.4.2). As in [6], it will be sufficient to demonstrate the relationship between the quality factors in the 1D case. We consider standing waves with real wavenumbers and propagating waves with real frequencies.

Experimental data on the application of Knopoff et al.'s [6] equation for surface waves based on group velocity can be found in Ben-Menahem [9] and Ben-Menahem and Sing ([10], Chapter 10). Previously, the theory was verified experimentally with surface waves [11]. Here we consider body waves. The theory could also be used to analyze the attenuation of edge waves (e.g. [12]). As mentioned above, the work of Knopoff et al. has hardly been considered in the literature. For instance, [10,13], Box 5.7, p. 162) do not report Knopoff et al. Eqs. (1) and (2) in their books, being A. Ben-Menahem and K. Aki co-authors of this paper.

2. Propagating wave

The equation of momentum conservation in 1D space is

$$\partial_x \sigma = \rho \ddot{u}, \qquad \text{or} \quad \partial_{xx} \sigma = \rho \ddot{\epsilon}, \tag{3}$$

where σ is the stress, ρ is the mass density, u is the displacement, ∂_x denotes a spatial derivative along the x-direction, such that $\epsilon = \partial_x u$ is the strain (e.g., [2], Eq. 2.81). An overdot denotes a time derivative.

We assume a propagating wave (see [2]; Section 2.3) whose wave kernel is

$$\exp[\mathrm{i}(\omega t - kx)], \quad k = \kappa - \mathrm{i}\alpha, \tag{4}$$

where ω is the angular frequency (a real quantity, as in [6]), k is the complex wavenumber, κ is the real wavenumber, α is the attenuation factor and $\mathrm{i} = \sqrt{-1}$.

The frequency-domain Kelvin–Voigt stress–strain relation is

$$\sigma = M\epsilon, \quad M = M_R + \mathrm{i}\omega\eta \tag{5}$$

([2], Section 2.4.2), where M_R is the relaxed modulus and η is the viscosity.

The complex velocity is

$$v_c = \sqrt{\frac{M}{\rho}}, \tag{6}$$

the phase velocity is

$$v_p = [\mathrm{Re}(v_c^{-1})]^{-1}, \tag{7}$$

$$\alpha = -\omega \mathrm{Im}(v_c^{-1}), \tag{8}$$

and the quality factor is

$$Q = \frac{\mathrm{Re}(v_c^2)}{\mathrm{Im}(v_c^2)} = \frac{1}{\omega\tau}, \quad \tau = \frac{\eta}{M_R}. \tag{9}$$

The energy velocity is

$$v_e = v_p \tag{10}$$

([14], Section 3.4.3; [2], Eq. 2.121).

The group velocity of the Kelvin–Voigt solid has been obtained by Casula and Carcione [15], Eqs. 13 and 18):

$$v_g = \frac{\partial_\omega}{\partial\kappa} = \left\{ \mathrm{Re}\left[\frac{1}{v_c}\left(1 - \frac{\omega}{2}\cdot\frac{M'}{M}\right)\right]\right\}^{-1} = \left\{ \mathrm{Re}\left[\frac{1}{v_c}\left(\frac{M_R + \frac{1}{2}\mathrm{i}\omega\eta}{M_R + \mathrm{i}\omega\eta}\right)\right]\right\}^{-1}, \tag{11}$$

where the prime indicates differentiation with respect to ω.

3. Standing wave

We assume a standing or stationary wave

$$\exp[\mathrm{i}(\Omega t - \kappa_T x)], \quad \Omega = \omega + \mathrm{i}\omega_I, \tag{12}$$

where Ω is the complex frequency and the wavenumber κ_T is real (as in [6]).

The dispersion equation is obtained by replacing (12) into (3):

$$\kappa_T^2 M_T = \rho \Omega^2. \tag{13}$$

and it can be shown that the complex modulus is

$$M_T = M_R + \mathrm{i}\Omega\eta = M_R + \mathrm{i}\omega\eta - \omega_I \eta. \tag{14}$$

Therefore, ω_I is the temporal attenuation factor.

Taking real and imaginary parts in Eq. (13) and solving for ω_I and κ_T, we obtain

$$\omega_I = \omega_0 - \sqrt{\omega_0^2 - \omega^2}, \quad \omega_0 = \frac{1}{\tau} = \frac{M_R}{\eta}, \tag{15}$$

such that $\omega_I = 0$ at $\omega = 0$ (lossless limit), and

$$\kappa_T = \sqrt{\frac{2\rho\omega_I}{\eta}}. \tag{16}$$

Unlike for propagating waves, the temporal attenuation factor ω_I (and Q) of the Kelvin–Voigt medium has a cut-off frequency given by ω_0 where Q vanish (total diffusion) (for $\omega > \omega_0$, ω_I becomes complex).

In this case ([2], Section 2.3.3)

$$v_{cT} = \frac{\Omega}{\kappa_T} = \sqrt{\frac{M_T}{\rho}} \tag{17}$$

$$v_{pT} = \frac{\omega}{\kappa_T} = \mathrm{Re}(v_{cT}), \tag{18}$$

$$\omega_I = \omega \cdot \frac{\mathrm{Im}(v_{cT})}{\mathrm{Re}(v_{cT})}, \tag{19}$$

$$Q_T = \frac{\mathrm{Re}(v_{cT}^2)}{\mathrm{Im}(v_{cT}^2)} = \sqrt{\left(\frac{\omega_0}{\omega}\right)^2 - 1} = \sqrt{Q^2 - 1}, \tag{20}$$

since, $Q = \omega_0/\omega$, and

$$v_{eT} = [\mathrm{Re}(v_{cT}^{-1})]^{-1}. \tag{21}$$

The temporal group velocity can be obtained by differentiation of Eq. (16):

$$v_{gT} = \frac{\partial_\omega}{\partial \kappa_T} = \omega^{-1}(\omega_0 - \omega_I)\sqrt{\frac{2\eta\omega_I}{\rho}}. \tag{22}$$

The standing-wave velocities that were formally defined do not have a precise physical meaning, since such a wave has no velocity. A standing wave is the sum of two out-of-phase propagating waves with the same frequencies moving in opposite directions.

On the basis of the previous equations, but an arbitrary stress–strain relation, an alternative analysis of Knopoff et al. [6] relations is given in the appendix.

4. Examples

The simplest case is the dispersionless model used by Carcione et al. [16] to perform seismic migration. Indeed, in the absence of dispersion there is no distortion of the wave packet and the group and phase velocities are the same, which means that the temporal and spatial quality factors coincide. The mathematics is as follows. The pressure wave equation is $\ddot{p} = c^2 \partial_{xx} p - 2\gamma \dot{p} - \gamma^2 p$, where p is the pressure and γ is a damping parameter. It is easy to show that $v_c = c/(1 - \mathrm{i}\gamma/\omega)$, $v_{cT} = c(1 + \mathrm{i}\gamma/\omega)$, $v_p = v_g = c$, $\omega_I = \gamma$, $\kappa = \kappa_T = \omega/c$ and $Q_T = Q = (\omega^2 - \gamma^2)/(2\gamma\omega)$.

Let us consider the Kelvin–Voigt solid and a reference frequency f_0 such that $\omega_0 = 2\pi f_0 = \tau_0^{-1}$ and the spatial quality factor at this frequency is

$$Q_0 = \frac{M_R}{\omega_0 \eta} = \frac{1}{\omega_0 \tau}, \quad \tau = \frac{1}{\omega_0 Q_0} \tag{23}$$

and

$$\eta = \tau M_R = \frac{M_R}{2\pi f_0 Q_0}. \tag{24}$$

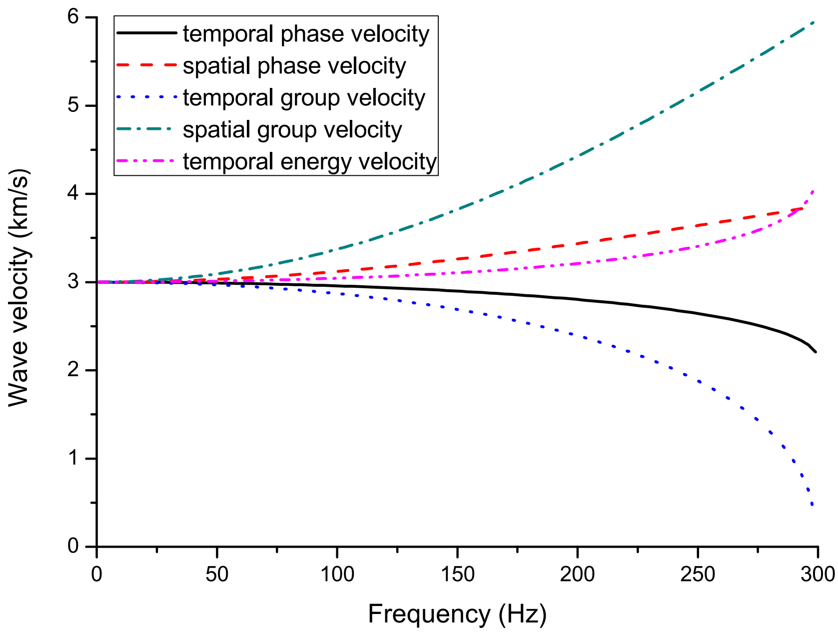

Fig. 1. Plot of v_{pT} (Eq. (18)), v_p (Eq. (7)), v_{gT} (Eq. (22)), v_g (Eq. (11)) and v_{eT} (Eq. (21)) as a function of frequency [$v_e = v_p$ (Eq. (10))].

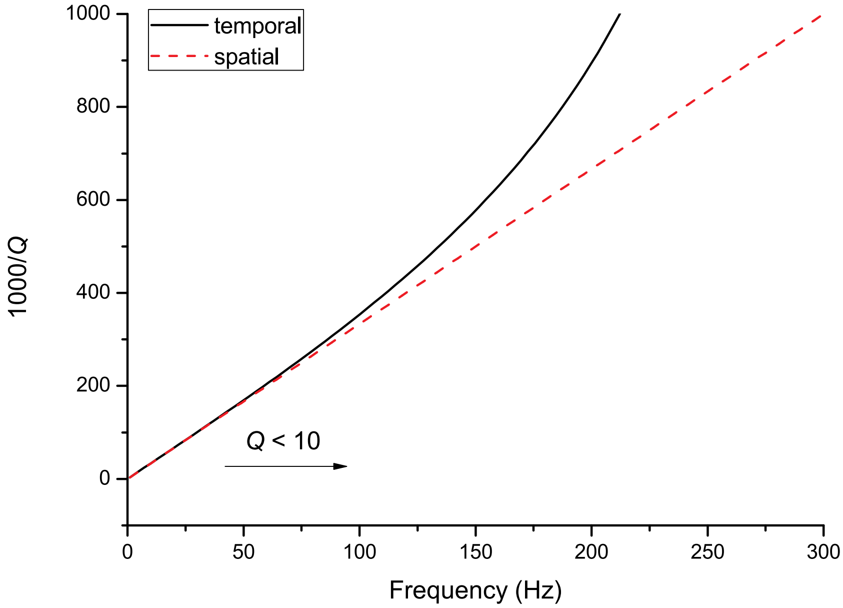

Fig. 2. Plot of $1000/Q_T$ (Eq. (20)) and $1000/Q$ (Eq. (9)) as a function of frequency.

Assume $\rho = 2400$ kg/m^3, $M_R = \rho c^2$, where $c = 3$ km/s, $f_0 = 300$ Hz and $Q_0 = 1$, such that Q is ∞ at zero frequency (lossless case) and decreases to Q_0 at f_0 (diffusion) [see Eq. (9)].

Figs. 1, 2 and 3 show the velocities, dissipation factors and attenuation factors as a function of frequency. Above $f = 30$ Hz, the spatial quality factor Q becomes less than 10 and the low-loss approximation does not hold. Below 30 Hz, all the quantities show similar values. At the "resonance" frequency f_0 the temporal values diverge.

As mentioned in the introduction, Knopoff et al. [6] have shown that the temporal and spatial quality factors are related by Eq. (1) and that $\omega_I = v_g \alpha$ for low-loss media; see [6], Eq. 7), where they indicate that this relation holds for high values of the quality factor. The curves are represented in Fig. 3 and show a good agreement, even for very low values of Q. Also represented (dotted line) is the spatial energy velocity (equal to the spatial phase velocity) times the spatial attenuation factor α. The spatial group velocity

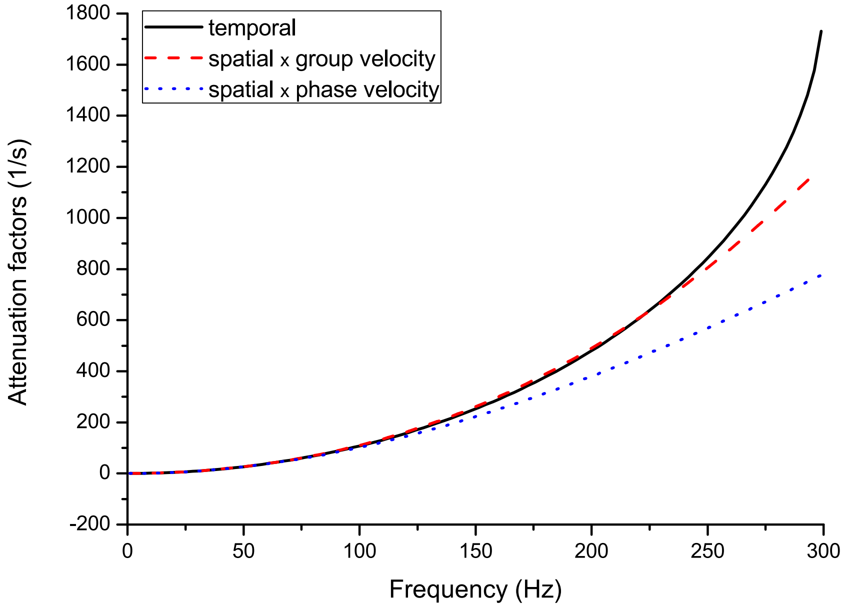

Fig. 3. Plot of ω_I (Eq. (19)), $v_g\alpha$ (Knopoff et al. relation) (Eqs. (8) and (11)) and $v_p\alpha$ (Eqs. (7) and (8)) as a function of frequency.

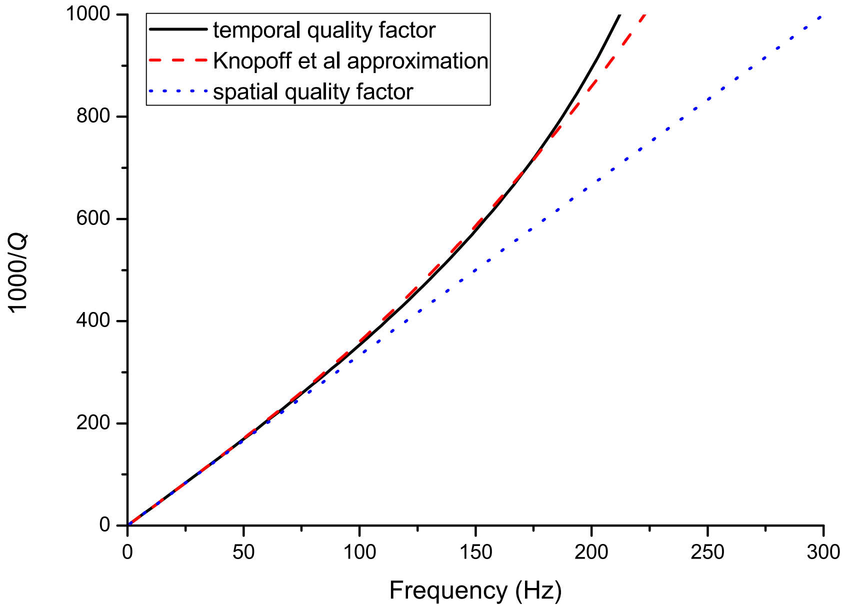

Fig. 4. Plot of Q_T, $(v_g/v_p)Q$ and $[v_p/(v_e = v_p)]Q = Q$ as a function of frequency.

performs better in the comparison. In the case of strong attenuation, numerical simulations could possibly be used to establish the relation between the two quality factors.

It is well known that the group velocity loses physical meaning in anelastic media (Ben-Menahem and Singh, Section 10.1.3; [2], Sections 2.10 (1-D case) and 4.6.5). Only for large values of Q the velocity v_g retains its physical meaning, and this is shown in Fig. 1. It is $v_e = v_p$ for propagating homogeneous waves in isotropic anelastic media. Despite this fact, we can see in Fig. 4 that the spatial group velocity performs better than the spatial energy velocity when we compare the spatial and temporal quality factors. The dashed lines correspond to $[v_p/(v_e = v_p)]Q = Q$. These calculations confirm the approximations of [6].

For further check, we now consider the perfectly constant-Q model introduced by [17]. The complex modulus of a propagating wave is

$$M(\omega) = \rho c^2 \left(\frac{\mathrm{i}\omega}{\omega_0}\right)^{2\nu}, \quad \nu = \frac{1}{\pi}\tan^{-1} Q_0^{-1} \tag{25}$$

([18], p. 54, Eq. 111; [2], Eq. 2.237), where ω_0 is a reference frequency, with $Q = Q_0$ at all frequencies. The phase and group velocities are

$$v_p = c\left[\cos\left(\frac{\pi\nu}{2}\right)\right]^{-1}\left|\frac{\omega}{\omega_0}\right|^{\nu} \quad \text{and} \quad v_g = \frac{v_p}{1-\nu} \tag{26}$$

[19]. Note that the correction $v_p/v_g = 1-\nu$ is constant as a function of frequency.

On the other hand, the complex modulus of a standing wave is

$$M_T(\omega) = \rho c^2 \left(\frac{\mathrm{i}\Omega}{\omega_0}\right)^{2\nu}. \tag{27}$$

The wave equation can be expressed in terms of fractional derivatives (see, [2], Eq. 2.254), from which the dispersion equation can be obtained, whose solutions are

$$\omega_I = \omega\tan\left(\frac{\pi\gamma}{2}\right), \quad \kappa_T = \frac{1}{c}\left[\frac{\omega\ \omega_0^{\gamma}}{\cos\left(\frac{\pi\gamma}{2}\right)}\right]^{1-\nu} \quad \gamma = \frac{\nu}{1-\nu}. \tag{28}$$

Let us assume $f_0 = 30$ Hz and $Q_0 = 20$. Then, $\nu = 0.016$ and $v_p/v_g = 0.984$. Fig. 5 shows the quality factors (a) and attenuation factors (b) as a function of frequency. The crosses correspond to Q_T and ω_I obtained from Eqs. (1) and (2), respectively (Knopoff et al. approximations), and the dashed line in (a) is the spatial quality factor. As can be seen the agreement of the crosses with Q_T is excellent. The approximation deteriorates if the condition $Q \gg 1$ is not fulfilled.

The proposed theory is important in determining the correction to be applied when comparing quality factors measured by different experiments, namely, standing waves generated in a confined sample (resonant and forced oscillations), quasi-static experiments [20] and waves propagating through the medium (ultrasonic methods, i.e., transmission and pulse-echo experiments) ([1], App. B).

Actually, the correction was first noticed in surface waves. Here, we have shown with examples that the correction applies to body waves. For surface waves, the correction can be even more significant. Ewing and Press [21] and Sato [22] used the phase velocity and not the group velocity in Eq. (2), and therefore their values of Q_T must be multiplied by the factor v_p/v_g. Brune [11] found that the factor v_p/v_g for Rayleigh waves varies from about 1.2 at 150 s to 1.4 at 350 s and 1.3 at 500 s (periods), while the factor for Love waves is about 1.2 at 300 s and 1.3 at 500 s.

5. Conclusions

Knopoff et al. relationship between the temporal and spatial quality factors has been verified using two stress–strain relationships, namely those of the Kelvin–Voigt model and the constant-Q model. The analysis provides explicit expressions for the quality factors, attenuation factors and wave velocities as a function of frequency. In particular, the theory has been applied to compare the quality factors for free vibration modes in unbounded media and propagating body waves. This work can be extended to test the relationship explicitly for normal vibration models, e.g. of the earth, and surface waves, such as Rayleigh and Love waves in anelastic media.

CRediT authorship contribution statement

José M. Carcione: Conceptualization, Methodology, Writing – original draft. **Jing Ba:** Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Knopoff relations

Let us assume that for standing waves,

$$\exp(-\omega_I t) = \exp(-\omega_I x/v_g), \tag{29}$$

where $v_g = x/t$ is the spatial group velocity. Equating (29) to the spatial damping $\exp(-\alpha x)$, we obtain Eq. (2).

On the other hand, since $v_{cT} = \Omega/\kappa_T$ and $Q_T = \mathrm{Re}(v_{cT}^2)/\mathrm{Im}(v_{cT}^2)$, we have

$$\omega_I^2 + 2\omega Q_T \omega_I - \omega^2 = 0. \tag{30}$$

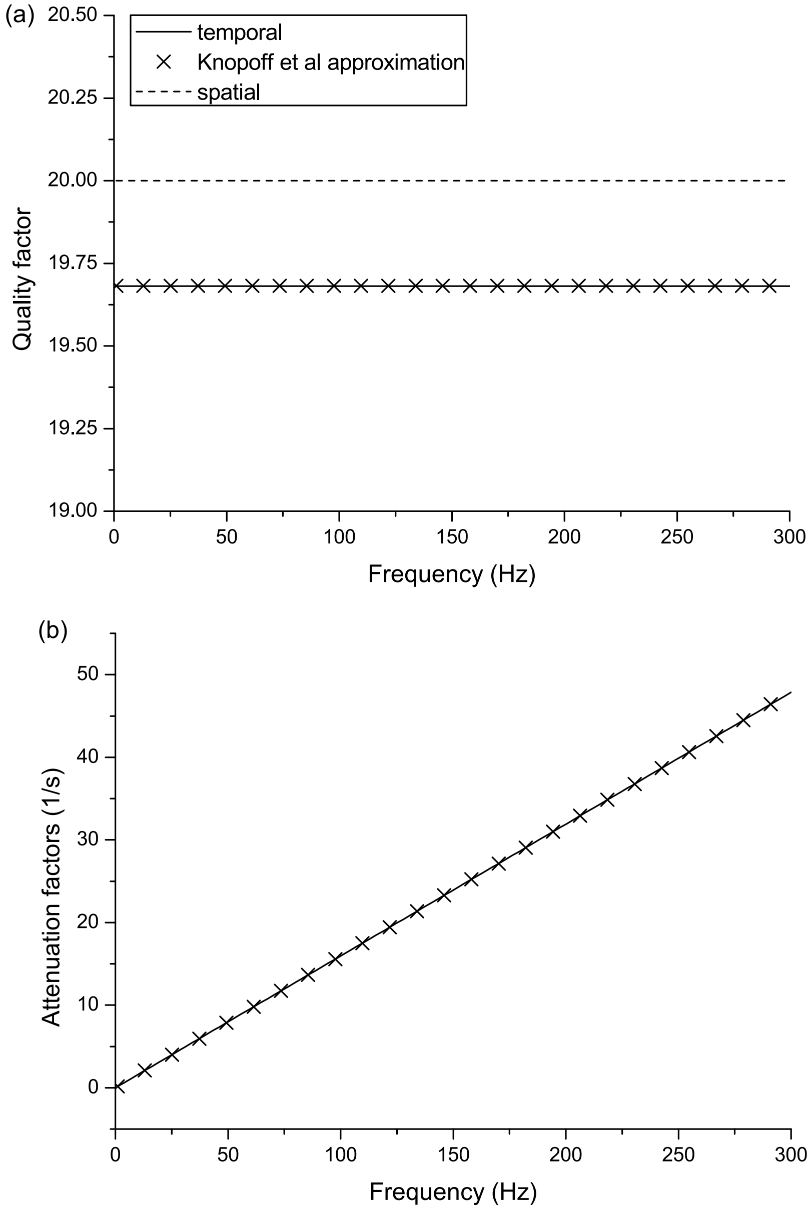

Fig. 5. Quality factors (a) and attenuation factors (b) as a function of frequency. The crosses correspond to Q_T and ω_I obtained from Eqs. (1) and (2), respectively (Knopoff et al. approximations), and the dashed line in (a) is the spatial quality factor Q. The solid lines represent Q_T (a) and ω_I (b).

Solving for ω_I, we obtain

$$\omega_I = \omega \left(\sqrt{Q_T^2 + 1} - Q_T \right), \tag{31}$$

or

$$\omega_I = \omega \left(Q_T \sqrt{1 + \frac{1}{Q_T^2}} - Q_T \right), \tag{32}$$

If $Q_T \gg 1$, a Taylor expansion gives

$$\omega_I \approx \omega \left[Q_T \left(1 + \frac{1}{2Q_T^2} \right) - Q_T \right] = \frac{\omega}{2Q_T} \tag{33}$$

or

$$Q_T \approx \frac{\omega}{2\omega_I}. \tag{34}$$

Then, equating (29) to the $\exp(-\alpha x)$, using (34) and the fact that for low-loss media, $\alpha = \omega/(2Qv_p)$, we obtain Eq. (1).

Data availability

Data will be made available on request.

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