The Kramers-Kronig relations and the analogy between electromagnetic and mechanical waves

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ABSTRACT The important consequence of the Kronig-Kramers relations (KKrs) is that dissipative behaviour in material media inevitably implies the existence of dispersion, i.e. a frequency dependence in the constitutive equations. Basically, the relations are the frequency-domain expression of causality and correspond mathematically to pairs of Hilbert transforms. The relations have many forms and can be obtained with diverse mathematical tools. Here, two different demonstrations are given in the electromagnetic case, illustrating the eclectic mathematical apparatus available for this purpose. Then, we apply the acoustic (mechanical)-electromagnetic analogy to obtain the elastic versions. One major consequence is wave propagation attenuation and pulse spreading, that is, the progressive widening of a pulse as it propagates through a medium [vacuum seems to be the only "medium" where this does not occur (electromagnetic dispersion), while mechanical waves do not propagate]. Therefore, we derive KKrs that relate the wave velocity to the attenuation and quality factors. Finally, we discuss the concepts of stability and passivity and provide a novel algorithm to compute the relations numerically by using the fast Fourier transform.

1. Introduction

Many scientists contributed to the wave theory, and since the beginnings there has been an interplay between the theory of light and mechanical wave propagation. As early as 1637 René Descartes provided and explanation of the rainbow and used Snell's law to study the reflection and refraction of light. In 1660, Robert Hooke formulated the stress-strain relation of solids and assumed that light propagates at a finite speed as oscillations of the medium. In the 19th century, Thomas Young was the first to consider shear as an elastic strain, and in 1809 Étienne-Louis Malus discovered polarisation of light by reflection. In 1821, Fresnel obtained the wave surface of an optically biaxial crystal, assuming that light waves are vibrations of the ether in which longitudinal vibrations of P waves do not propagate (Carcione and Helbig, 2008). He showed that if light were a transverse wave, then it would be possible to develop a theory accommodating the polarisation of light. George Green made extensive use of the analogy between elastic waves and light waves, and later James Clerk Maxwell and Lord Kelvin used physical and mathematical analogies to study wave phenomena in elastic theory and electromagnetism, when light was discovered to be oscillations of electric and magnetic fields. In fact, the displacement current introduced by Maxwell into the electromagnetic equations arises from the analogy with elastic displacements.

Maxwell assumed his equations were valid in an absolute system regarded as a medium (called the ether) that filled the whole space. The ether was in a state of stress, and would only transmit transverse waves, but with the advent of the relativity theory the concept of the ether was abandoned. However, the fact that electromagnetic waves are transverse waves is important. Carcione and Cavallini (1995) showed that the 2D Maxwell equations describing propagation of the TM mode in anisotropic media is mathematically equivalent to the SH wave equation in an anisotropic-viscoelastic solid where attenuation is described with the Maxwell model.

Any medium (unless vacuum), natural or man-made, dissipates energy when subject to electromagnetic and mechanical fields. Dielectric or magnetic relaxation may occur under the influence of electromagnetic fields. Electric displacement and electric field are related by a relaxation function and a complex dielectric permittivity in the time and frequency domains, respectively (Böttcher and Bordewijk, 1996; Orfanidis, 2016). In viscoelasticity, stress and strain are related by a relaxation function or a complex stiffness modulus (Nowick and Berry, 1972; Christensen, 1982). According to Carcione and Cavallini (1995), one common analogy is given by the fact that for dielectric relaxation, electric field strength is equivalent to elastic stress, electric displacement to elastic strain, and the dielectric constant to elastic compliance. For magnetic relaxation, the corresponding variables are the magnetic field strength, intensity of magnetisation, and magnetic susceptibility, respectively. For instance, Carcione and Schoenberg (2000) used these equivalences to introduce viscoelastic relaxation functions and simulate electromagnetic fields. However, the mathematical analogy is not complete, since differences arise from the tensor nature of stress and strain as compared with the vector nature of the electromagnetic fields.

Proper models should satisfy the Kramers-Kronig relations (KKRs), known from the beginning of the 20th century from the works of de Laer Kronig (1926) and Kramers (1927) on electromagnetism, showing the interrelation between the real and imaginary parts of the complex susceptibility. Electrical and mechanical representations include the Debye model, used to describe the behaviour of dielectric materials, and the Zener viscoelastic model, respectively, both being mathematically equivalent (Carcione, 2014). In viscoelasticity, the KKRs connect the real and imaginary parts of the stiffness modulus. Carcione *et al*. (2019) provide a complete derivation of the relations using the Sokhotski-Plemelj equation, showing explicitly what are the conditions for the relations to hold.

There are many forms and ways to obtain the KKrs. Here, two different demonstrations are given, illustrating the eclectic mathematical apparatus available. Then, we apply the acoustic-electromagnetic analogy to obtain the elastic versions. In particular, it is of interest the effects caused by wave propagation, which induces velocity dispersion and attenuation. In this sense, we derive KKrs that related the wave velocity to the attenuation and quality factors. Other applications of the KKrs include the magnetic permeability (Silveirinha, 2011), circuits (electrochemical impedance) (Esteban and Orazem, 1991), electric circuits (Guillemin, 1949), optical spectroscopy (Lucarini *et al*., 2005) and quantum mechanics (Longhi, 2017).

2. The Kramers-Kronig relations

2.1. Dielectric permittivity

Generally, the electric field **E** is defined as the field in vacuum plus a polarisation due to the presence of matter. The time-domain permittivity is:

$$\epsilon(t) = \epsilon_0 H(t) + \chi(t), \tag{1}$$

where ϵ_0 is the permittivity of vacuum, χ is the electric susceptibility and H is the Heaviside function. The electric displacement is (e.g. Orfanidis, 2016):

$$\mathbf{D} = \epsilon * \dot{\mathbf{E}} = \dot{\epsilon} * \mathbf{E} = \epsilon_0 \mathbf{E} + \dot{\chi} * \mathbf{E} \tag{2}$$

since $\dot{H} = \delta$ (Dirac's function), where '*' denotes time convolution and a dot above a variable time differentiation. The second term in Eq. 2 is the polarisation.

Here, we do not make a distinction between the vacuum term and the polarisation and consider the permittivity. Let us define:

$$\hat{\epsilon}(t) = \epsilon(t) - \epsilon^\infty \geq 0, \tag{3}$$

such that

$$\epsilon^\infty = \epsilon(t = 0) \text{ and } \hat{\epsilon}(t) \to 0 \text{ when } t \to 0, \tag{4}$$

where ϵ^∞ is the optical (high-frequency) permittivity.

From Eq. 2, we may write

$$\mathbf{D}(t) = \int_{-\infty}^{t} \epsilon(t - t')\dot{\mathbf{E}}(t')dt' = -\int_{0}^{\infty} \epsilon(\tau)\dot{\mathbf{E}}(t - \tau)d\tau, \tag{5}$$

because $\epsilon(t)$ is causal and we defined $\tau = t - t'$ (the time derivatives are calculated with respect to the arguments). Evidencing explicitly the instantaneous response,

$$\mathbf{D}(t) = \epsilon^\infty \mathbf{E}(t) - \int_{0}^{\infty} \hat{\epsilon}(\tau)\dot{\mathbf{E}}(t - \tau)d\tau, \tag{6}$$

where we used Eq. 3.

Now, substitute a Fourier component for the electric field $\mathbf{E}(t) = \mathbf{E}_0 \exp(i\omega t)$, to obtain:

$$\mathbf{D}(t) = \epsilon^\infty \mathbf{E}(t) + i\omega \mathbf{E}_0 \int_{0}^{\infty} \hat{\epsilon}(\tau) \exp[i\omega(t - \tau)]d\tau, \tag{7}$$

where i = $\sqrt{-1}$.

The Fourier transform of Eq. 2 gives:

$$\mathcal{F}[\mathbf{D}(t)] = \epsilon(\omega)\mathcal{F}[\mathbf{E}(t)], \tag{8}$$

where

$$\epsilon(\omega) = \mathcal{F}[\dot{\epsilon}(t)] = \int_{-\infty}^{\infty} \dot{\epsilon}(t) \exp(-i\omega t) dt \tag{9}$$

is the complex permittivity (for clarity, we use the same symbol in both Fourier domains). Since

$$\epsilon(\omega) = \epsilon_1 + i\epsilon_2 = \mathbf{D}/\mathbf{E} = \epsilon^{\infty} + i\omega \int_0^{\infty} \hat{\epsilon}(\tau) \exp(-i\omega\tau) d\tau, \tag{10}$$

we have:

$$\epsilon_1(\omega) = \epsilon^{\infty} + \omega \int_0^{\infty} \hat{\epsilon}(t) \sin(\omega t) dt. \tag{11}$$

and

$$\epsilon_2(\omega) = \omega \int_0^{\infty} \hat{\epsilon}(t) \cos(\omega t) dt. \tag{12}$$

Eq. 12 is a cosine transform, whose reverse transformation is:

$$\hat{\epsilon}(t) = \frac{2}{\pi} \int_0^{\infty} \frac{\epsilon_2(\omega')}{\omega'} \cos(\omega' t) d\omega'. \tag{13}$$

Substituting Eq. 13 into Eq. 11 and re-ordering terms yields:

$$\epsilon_1(\omega) = \epsilon^{\infty} + \frac{2\omega}{\pi} \int_0^{\infty} \frac{\epsilon_2(\omega')}{\omega'} \left[\int_0^{\infty} \sin(\omega t) \cos(\omega' t) dt \right] d\omega'. \tag{14}$$

we have:

$$\int_0^{\infty} \sin(\omega t) \cos(\omega' t) dt = \frac{1}{2} \int_0^{\infty} \left\{ \sin[(\omega + \omega')t] + \sin[(\omega - \omega')t] \right\} dt$$

$$= -\frac{1}{2} \left| \frac{\cos[(\omega + \omega')t]}{\omega + \omega'} + \frac{\cos[(\omega - \omega')t]}{\omega - \omega'} \right|_0^{\infty} \tag{15}$$

$$= \frac{1}{2} \left[\frac{1}{\omega + \omega'} + \frac{1}{\omega - \omega'} \right] = \frac{\omega}{\omega^2 - \omega'^2},$$

where the integrals have been handled as

$$\lim_{\beta \to 0} \int_0^\infty \exp(-\beta t) \sin[(\omega \pm \omega')t]dt \tag{16}$$

in the second equality, to warrant convergence at the upper limit [a more rigorous proof, based on complex-variable theory, is given in §123 of Landau and Lifschitz (1958)].

Then, substituting Eq. 15 into Eq. 14 gives:

$$\epsilon_1(\omega) = \epsilon^\infty + \frac{2}{\pi} \int_0^\infty \frac{\omega^2 \epsilon_2(\omega')}{\omega'(\omega^2 - \omega'^2)} d\omega'. \tag{17}$$

Because of the Hermitian properties of $\epsilon(\omega)$ (the permittivity function is real valued) (Bracewell, 2000), we have:

$$\epsilon_1(\omega) = \epsilon_1(-\omega), \quad \epsilon_2(\omega) = -\epsilon_2(-\omega), \tag{18}$$

indicating that ϵ_1 and ϵ_2 are even and odd functions of ω, respectively.

Then, the integrand in Eq. 17 is an even function and we have:

$$\epsilon_1(\omega) = \epsilon^\infty + \frac{1}{\pi} \int_{-\infty}^\infty \frac{\omega^2 \epsilon_2(\omega')}{\omega'(\omega^2 - \omega'^2)} d\omega'. \tag{19}$$

Since

$$\frac{\omega^2}{\omega'(\omega^2 - \omega'^2)} = \frac{\omega}{\omega'(\omega - \omega')} - \frac{\omega}{\omega^2 - \omega'^2}, \tag{20}$$

we have:

$$\epsilon_1(\omega) = \epsilon^\infty + \frac{1}{\pi} \int_{-\infty}^\infty \frac{\omega \epsilon_2(\omega')}{\omega'(\omega - \omega')} d\omega' - \frac{1}{\pi} \int_{-\infty}^\infty \frac{\omega \epsilon_2(\omega')}{\omega^2 - \omega'^2} d\omega'. \tag{21}$$

The second integral is zero, because ϵ_2 is odd. Then:

$$\epsilon_1(\omega) = \epsilon^\infty + \frac{1}{\pi} \int_{-\infty}^\infty \frac{\omega \epsilon_2(\omega')}{\omega'(\omega - \omega')} d\omega'. \tag{22}$$

We can further simply Eq. 19. Since:

$$\frac{\omega}{\omega'(\omega - \omega')} = \frac{1}{\omega'} + \frac{1}{\omega - \omega'}, \tag{23}$$

we have:

$$\epsilon_1(\omega) = \epsilon^\infty + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon_2(\omega')}{\omega'} d\omega' + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon_2(\omega')}{\omega - \omega'} d\omega'.$$ (24)

From Eq. 13, the first integral is $\hat{\epsilon}(t = 0)$ since $\epsilon_2(\omega')/\omega'$ is an even function. Using Eqs. 3 and 4:

$$\hat{\epsilon}(t = 0) = \epsilon(t = 0) - \epsilon^\infty = 0.$$ (25)

Then:

$$\epsilon_1(\omega) = \epsilon^\infty + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon_2(\omega')}{\omega - \omega'} d\omega'$$ (26)

is another expression of the KKR (Böttcher and Bordewijk, 1996).

Polarisation can be dielectric and orientational. The first occurs when a dipole moment is formed in an insulating material due to an externally applied electric field, while orientational polarisation arises when there is a permanent dipole moment in the material. In Böttcher (1993), ϵ^∞ is the dielectric constant at a sufficiently high frequency, when the permanent dipoles (i.e. the orientational polarisation) can no longer follow the changes of the field. As stated by Böttcher and Bordewijk (1996), Eq. 26 holds for the case that the induced polarisation follows the field without delay, implying that the integration should be cut off at a frequency in the range between the characteristic frequencies of the orientational and the induced polarisation. If the integrations are performed over the whole frequency range, one implicitly accounts for the fact that the intramolecular motions connected with the induced polarisation are not infinitely fast, and one should take ϵ_0 instead of ϵ^∞ in the equation, yielding:

$$\epsilon_1(\omega) = \epsilon_0 + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon_2(\omega')}{\omega - \omega'} d\omega'.$$ (27)

This corresponds to a KKr for the susceptibility $\chi(\omega) = \epsilon(\omega) - \epsilon_0$, according to Eq. 1 (e.g. Orfanidis, 2016), since $\chi_1 = \epsilon_1 - \epsilon_0$ and $\chi_2 = \epsilon_2$.

Following the same procedure for the imaginary part of the permittivity, we obtain:

$$\epsilon_2(\omega) = -\frac{\omega}{\pi} \int_{-\infty}^{\infty} \frac{[\epsilon_1(\omega') - \epsilon^\infty]}{\omega'(\omega - \omega')} d\omega',$$ (28)

and the simplified equation is

$$\epsilon_2(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon_1(\omega')}{\omega - \omega'} d\omega',$$ (29)

since $\epsilon_2(\omega = \infty) = 0$.

Eqs. 22 and 28 and Eqs. 26 and 29 are two equivalent Hilbert transform pairs (Bracewell, 2000), defining two different expressions of the KKRs. The Cauchy principal value of the improper integrals is intended in these calculations.

Defining the Hilbert transform of a function g as:

$$\mathcal{H}[g(\omega)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\omega')}{\omega' - \omega} d\omega', \tag{30}$$

Eqs. 26 and 29 read:

$$\epsilon_1 - \epsilon^{\infty} = -\mathcal{H}[\epsilon_2(\omega)] \quad \text{and} \quad \epsilon_2 = \mathcal{H}[\epsilon_1(\omega)], \tag{31}$$

respectively.

2.2. Alternative demonstration

Since $\hat{\epsilon}(t)$ (Eq. 4) is real, $\hat{\epsilon}(\omega)$ is Hermitian; that is:

$$\hat{\epsilon}_1(\omega) = \hat{\epsilon}_1(-\omega), \quad \hat{\epsilon}_2(\omega) = -\hat{\epsilon}_2(-\omega). \tag{32}$$

Furthermore, $\hat{\epsilon}$ can split into even and odd functions of time, $\hat{\epsilon}_e$ and $\hat{\epsilon}_o$, respectively, as

$$\hat{\epsilon}(t) = \frac{1}{2}[\hat{\epsilon}(t) + \hat{\epsilon}(-t)] + \frac{1}{2}[\hat{\epsilon}(t) - \hat{\epsilon}(-t)] \equiv \hat{\epsilon}_e + \hat{\epsilon}_o. \tag{33}$$

Since $\hat{\epsilon}$ is causal, $\hat{\epsilon}_o(t) = \operatorname{sgn}(t)\hat{\epsilon}_e(t)$, and

$$\hat{\epsilon}(t) = [1 + \operatorname{sgn}(t)]\hat{\epsilon}_e(t), \tag{34}$$

whose Fourier transform is:

$$\mathcal{F}[\hat{\epsilon}(t)] = \hat{\epsilon}(\omega) = \epsilon_1(\omega) - \epsilon^{\infty} - \left(\frac{i}{\pi\omega}\right) * [\epsilon_1(\omega) - \epsilon^{\infty}], \tag{35}$$

because $\mathcal{F}[\hat{\epsilon}_e] = \epsilon_1 - \epsilon^{\infty}$ and $\mathcal{F}[\operatorname{sgn}(t)] = 2/(i\omega)$ (Bracewell, 2000). Eq. 35 and the fact that $(\omega - \omega')^{-1}$ is ω-symmetric and odd imply:

$$\epsilon_2(\omega) = -\left(\frac{1}{\pi\omega}\right) * [\epsilon_1(\omega) - \epsilon^{\infty}] = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon_1(\omega')d\omega'}{\omega - \omega'}, \tag{36}$$

i.e. Eq. 29 (alternatively, one could also used the property that the Hilbert transform of a constant is zero).

Similarly, since $\hat{\epsilon}_e(t) = \text{sgn}(t)\hat{\epsilon}_o(t)$, it is $\hat{\epsilon}(t) = [\text{sgn}(t)+1]\hat{\epsilon}_o(t)$, and because $\mathcal{F}[\hat{\epsilon}_o] = \text{i}e_2$, we obtain:

$$\epsilon_1(\omega) - \epsilon^\infty = \left(\frac{1}{\pi\omega}\right) * \epsilon_2(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\epsilon_2(\omega')d\omega'}{\omega - \omega'}, \qquad (37)$$

i.e. Eq. 26.

In mathematical terms, $\epsilon_1 - \epsilon^\infty$ and ϵ_2 are Hilbert transform pairs. Causality also implies that ϵ has no poles (or is analytic) in the lower half complex ω-plane (Golden and Graham, 1988). A detailed and complete mathematical treatment of the KKrs is given in King (2009).

Including the ionic conductivity into the permittivity infringes the KKrs, Carcione (2014) shows that a generalised conductivity can be represented by the Kelvin-Voigt model, which does not satisfy the relations, because it does not meet the stability condition (Carcione *et al.*, 2019). For example, if the medium is conducting at zero frequency, $\epsilon - \text{i}\sigma/\omega$ is singular at $\omega = 0$. Thus, the KKrs are retained if we subtract this singularity, where σ > 0 is the zero frequency conductivity. Then, the KK-transforms between the real and imaginary parts of the effective permittivity are satisfied only if the imaginary part does not contain any conductivity. This fact has been exploited to retrieve the conductivity from that effective complex permittivity (Bakry and Klinkenbusch, 2018). Expressions of the KKrs for conductors are given in Eqs. 4.8 and 4.9 of Lucarini *et al.* (2005).

3 Acoustic-electromagnetic analogy

A brief comment on the analogy regarding the KKrs (without equations) is given in Gross (1975). The vector analogy between the acoustic and electromagnetic equations, based on the Maxwell mechanical model (viscoelasticity) and Maxwell's equations, has been first established by Carcione and Cavallini (1995), by which elastic SH waves are mathematically equivalent to transverse-magnetic (TM) waves. In this particular case:

$$
\begin{aligned}
\text{magnetic field} \quad &\Leftrightarrow \quad \text{particle velocity} \\
\text{electric displacement} \quad &\Leftrightarrow \quad \text{elastic strain} \\
\text{electric field} \quad &\Leftrightarrow \quad \text{elastic stress} \\
\text{magnetic permeability } (\mu) \quad &\Leftrightarrow \quad \text{mass density } (\rho) \\
\epsilon^\infty \quad &\Leftrightarrow \quad J_U \\
\sigma \quad &\Leftrightarrow \quad \eta^{-1} \\
\epsilon(\omega) = \epsilon^\infty - \frac{\text{i}}{\omega}\sigma \quad &\Leftrightarrow \quad J(\omega) = J_U - \frac{\text{i}}{\omega\eta},
\end{aligned}
\qquad (38)
$$

where σ is the electrical conductivity, *J* is the creep compliance, and J_U and η are the unrelaxed compliance and viscosity, respectively. The compliance may correspond to bulk or to shear deformations. The equivalence In Eq. 38 involves also the ionic-conductivity term of Maxwell's

equation, which has been included in the permittivity, i.e. we have considered **D+σE** instead of **D**. Other specific analogies are illustrated in Carcione and Robinson (2002), for the reflection-transmission problem, and in Carcione (2014).

For the purpose of this work, we need the following equivalences:

$$\epsilon(\omega) \quad \Leftrightarrow \quad J(\omega) \tag{39}$$

$$\epsilon^0 \quad \Leftrightarrow \quad J_R \tag{40}$$

$$\epsilon^\infty \quad \Leftrightarrow \quad J_U, \tag{41}$$

where J_R is the relaxed compliance.

Exploiting the analogy, we easily obtain KKrs equivalent to Eqs. 26 and 29:

$$J_1(\omega) = J_U + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{J_2(\omega')}{\omega - \omega'} d\omega' = J_U - \mathcal{H}[J_2(\omega)] \tag{42}$$

and

$$J_2(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{J_1(\omega')}{\omega - \omega'} d\omega' = \mathcal{H}[J_1(\omega)], \tag{43}$$

respectively. These two equations can also be deduced from Eqs. 2.4-3 and 2.4-4 of Nowick and Berry (1972) after some calculations, using the property $\omega'/(\omega'^2 - \omega^2) = 1/(\omega' - \omega) - \omega/(\omega'^2 - \omega^2)$ and the fact that $J_2(\omega)$ is an odd function.

Equivalent KKrs can be obtained for the complex stiffness or modulus $M = M_1 + iM_2 = 1/J$, related to the relaxation function:

$$M_1(\omega) = M_U + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{M_2(\omega')}{\omega - \omega'} d\omega' = M_U - \mathcal{H}[M_2(\omega)] \tag{44}$$

and

$$M_2(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{M_1(\omega')}{\omega - \omega'} d\omega' = \mathcal{H}[M_1(\omega)]. \tag{45}$$

These equations have been shown to be equivalent to Eqs. 2.4-6 and 2.4-7 of Nowick and Berry (1972) by Carcione *et al.* (2022).

3.1. Wave velocity and attenuation

The KKRs can be applied to wave velocity and attenuation, which is useful in seismology. Let us define the complex wave velocity as

$$v_c(\omega) = \frac{1}{\sqrt{\mu\epsilon(\omega)}} \quad \Leftrightarrow \quad \frac{1}{\sqrt{\rho J(\omega)}} = \sqrt{\frac{M(\omega)}{\rho}}, \tag{46}$$

such that the complex slowness is:

$$\frac{1}{v_c} = \frac{1}{v_p} - \frac{i\alpha}{\omega}, \tag{47}$$

where v_p is the phase velocity and α is the attenuation factor (e.g. Carcione, 2014).

By virtue of the analogy, we may consider the elastic Eqs. 44 and 45. Let us identify $1/v_c$ with M, $1/v_p$-$1/v_\infty$ with M_1-M_U and $-\alpha/\omega$ with M_2, where $v_\infty = v_p(\omega = \infty)$ is the unrelaxed velocity. Performing similar mathematical developments to obtain those equations, we get:

$$\frac{1}{v_p(\omega)} - \frac{1}{v_\infty} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha(\omega')d\omega'}{\omega'(\omega - \omega')} \tag{48}$$

and

$$\alpha(\omega) = \frac{\omega}{\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{v_p(\omega')(\omega - \omega')}, \tag{49}$$

which are Eqs. 13 in Box 5.8 in Aki and Richards (2009), where we used that the Hilbert transform of a constant is zero. These equations hold in the electromagnetic case if $v_\infty = 1/\sqrt{\mu\epsilon^\infty}$ is the optical velocity.

Using Eq. 23, we easily get:

$$\omega\left(\frac{1}{v_p(\omega)} - \frac{1}{v_\infty}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha(\omega')d\omega'}{(\omega' - \omega)} = \mathcal{H}[\alpha(\omega)] \tag{50}$$

and

$$\alpha(\omega) = -\omega\mathcal{H}\left[\frac{1}{v_p(\omega')}\right]. \tag{51}$$

The first is Eq. 3 in Box 5.8 in Aki and Richards (2009).
Similar relations between velocity and quality factor can easily be obtained by considering that

$$\mathcal{Q} = \frac{|\omega|}{2\alpha v_p}, \tag{52}$$

defined as the total energy divided by the dissipated energy on a cycle of wave oscillation (Carcione, 2014). Then, the KKrs of Eqs.48 and 49 become:

$$\frac{1}{v_p(\omega)} - \frac{1}{v_\infty} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\omega'|}{\omega'v_p(\omega')\mathcal{Q}(\omega')}\frac{d\omega'}{\omega - \omega'} \tag{53}$$

and

$$\frac{1}{\mathcal{Q}(\omega)} = \frac{2\omega v_p(\omega)}{\pi |\omega|} \int_{-\infty}^{\infty} \frac{d\omega'}{v_p(\omega')(\omega - \omega')}. \tag{54}$$

Carcione *et al*. (2019) shows that the Maxwell and Zener model satisfy the KKRs but the Kelvin-Voigt model do not. If the medium is dispersive but lossless, $\mathcal{E}_1(\omega) = e^0$ or $J_1(\omega)$ - J_R can depend on ω through functions of ω whose Hilbert transform involves delta functions, which do not represent damping due to their zero bandwidth. In electromagnetism, a specific Lorenz model (e.g. Carcione *et al*., 2010) satisfies the relations, having zero dissipation in its resonances, where the real permittivity takes positive and negative values for certain frequencies even though the imaginary part is zero (Poon and Francis, 2009; Orfanidis, 2016). On the other hand, lossless dispersion occurs at frequencies is far away from the resonances, where the energy and group velocities coincide (Carcione *et al*., 2010; Orfanidis, 2016).

An example of dispersionless lossy medium is given by a complex velocity $v_C = \omega c / (\omega - i\gamma)$, where c is a constant velocity and γ a damping factor (Carcione *et al*., 2016). It is $v_P = c$ and $\alpha = \gamma/c$. However, it can be seen from Eq. 49 that this medium does not satisfy the KKRs.

3.2. Stability and passivity

The concept of stability is related to the boundedness of the response. A strong condition is square-integrability of $M(\omega)$ along the real axis of the ω-plane, which implies:

$$\int_{-\infty}^{\infty} |M(\omega)|^2 d\omega < C, \tag{55}$$

where *C* is a constant (e.g. Nussenzveig, 1972). This is equivalent to $M(\omega) \to 0$, for $|\omega| \to \infty$ $(\pi \geq \arg(\omega) \geq 0)$. Generally, this condition cannot be satisfied, but rather the weaker one that $|M(\omega)|$ is bounded, i.e. $|M(\omega)|^2 < C$ is verified. A lossless medium and the Maxwell and Zener models satisfy the condition, but the Kelvin-Voigt and constant-*Q* models do not. In fact, in the case of the Zener model, *M* satisfies the weak condition and $M - M(\infty)$ is square integrable. A constant-*Q* model has $M(\omega) \propto \omega^{2\gamma}$, where 0<γ<1/2 (Carcione, 2014) and does not satisfy the conditions (Carcione *et al*., 2019).

Passivity is another characteristic of a causal system by which it can only absorb and not generate energy. Nussenzveig (1972) shows that that any linear passive system is causal, so that passivity is a stronger condition. In the context of electromagnetism, the condition is (Glasgow *et al*., 2001):

$$\mathrm{Im}[\epsilon(\omega)] < 0, \quad \omega > 0 \tag{56}$$

(we use the opposite sign convention for the Fourier transform). For viscoelasticity, the condition is Im[*J*(ω)] > 0, according to the acoustic- electromagnetic analogy, and Im[*M*(ω)=1/*J*(ω)] > 0 (conditions that can be checked with the Zener model in Eq. 59 below).

4. Numerical evaluation of the Kramers-Kronig relations

It is well known that the imaginary part of a complex trace z is the Hilbert transform of its real part (e.g. Cohen, 1995) i.e.:

$$z = z_1 + \mathrm{i}z_2 = z_1 + \mathrm{i}\mathcal{H}[z_1].$$
(57)

Denoting by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the forward and inverse Fourier operators, it is

$$z = \mathcal{F}^{-1}[Z], \quad Z = \begin{cases} Z_1 & \text{for 0 argument,} \\ 2Z_1 & \text{for positive arguments,} \\ 0 & \text{for negative arguments,} \end{cases}$$
(58)

where $Z_1 = \mathcal{F}[z_1]$ (Cohen, 1995). Then, Im[z] is the Hilbert transform of z_1.

Let us consider, as an example, Eq. 50 to compute the phase velocity from the attenuation factor and set $z_1 = \alpha$ and $z_2 = \omega(1/v_p - 1/v_\infty)$. The workflow is:

1. compute $Z_1 = \mathcal{F}[\alpha]$;   ;

2. compute Z as in Eq. 58;

3. compute $z = \mathcal{F}^{-1}[Z]$;

4. set $\omega\left(\dfrac{1}{v_p} - \dfrac{1}{v_\infty}\right) = z_2 = \mathcal{H}[z_1] = \mathrm{Im}\ [z]$;

5. obtain $v_p = \left(\dfrac{1}{v_\infty} + \dfrac{\mathrm{Im}[z]}{\omega}\right)^{-1}$.

Let us assume the Zener model, whose complex modulus is:

$$M(\omega) = \rho v_\infty^2 \cdot \frac{Q_0\left(\sqrt{Q_0^2 + 1} + 1\right)^{-1} + \mathrm{i}\omega\tau_0}{Q_0\left(\sqrt{Q_0^2 + 1} - 1\right)^{-1} + \mathrm{i}\omega\tau_0},$$
(59)

where Q_0 is the minimum quality factor at $\omega_0 = 2\pi f_0 = 1/\tau_0$. The high-frequency limit corresponds to the elastic case, with $M \to \rho v_\infty^2$. The complex velocity is given by Eq. 46 and the phase velocity and attenuation factor can be obtained from Eq. 47. We assume that the attenuation is known, i.e.:

$$\alpha = -\omega\mathrm{Im}\left[\frac{1}{v_c}\right]$$
(60)

and compute v_p using the preceding algorithm.

The Fourier transform is computed with the fast Fourier transform (FFT) (e.g. Carcione, 2014).
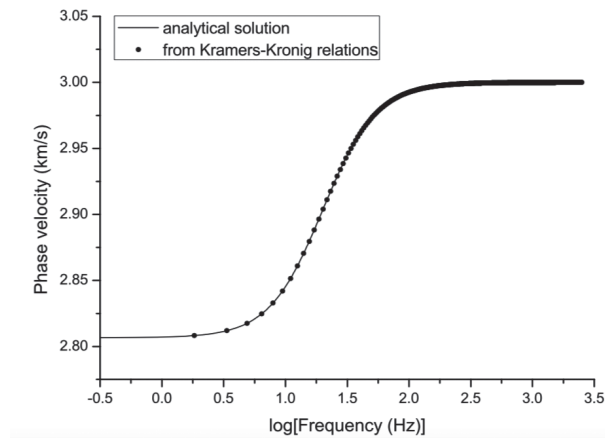
Fig. 1 - Analytical (solid line) and numerical (dots) phase velocities of the Zener (standard - linear solid) model. The latter is computed by numerically solving the Kramers-Kronig relations.

The arguments (frequencies) are defined as in Eq. 9.70 of Carcione (2014) and the attenuation array is the real part of a complex vector of length equal to a power of two, i.e. $n = 2^{14}$. We consider $v_\infty = 3 \ \mathrm{km/s}$, $Q_0 = 15$ and f_0 = 20 Hz. Fig. 1 compares the theoretical and numerical phase velocities, where we can observe that the agreement is excellent.

A generalisation of this example to the case of the fractional Zener model can be based on the work of Pritz (1999), to analyse the performance of the present algorithm as a function of the order of the fractional derivative.

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