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REVIEW PAPER

Seismic Q revisited

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Abstract
This paper revisits the theories of Q (quality factor) as a measure of the attenuation and velocity dispersion of a wave field. Q is a dimensionless measure of energy loss per cycle, and a proper understanding is important in a variety of fields, from seismology, geophysical prospecting to acoustics of materials. Measurements for standing modes and propagating waves differ and yield the temporal and spatial Q, respectively. This distinction is largely ignored in the literature. The relationship between these Qs is investigated for a power-law stress–strain relation based on spatial fractional derivatives that describes the behavior of compressional waves when combined with the conservation of momentum equation. In addition, the relationship between the quality factors for low-loss media proposed by Knopoff et al. 60 years ago is verified.

Keywords Temporal Q · Spatial Q · Phase velocity · Attenuation factor · Power-law stress–strain relation

Introduction

Seismic attenuation has a decisive influence on the signal and is therefore important for the synthesis of the waveform. It is parameterized by the quality factor Q, which accounts for the total energy loss of the seismic pulse during propagation and is a key parameter for extracting information about the subsurface layers (e.g., Gurevich and Carcione 2022; Qadrouh et al. 2016, 2018, 2020).

The distinction between spatial and temporal quality factors, Q and Q_T, is important in seismology, where the anelastic properties of free oscillations of the Earth (or normal modes) strictly differ from those of teleseismic waves. A relationship between these two different Q's and between the corresponding quality factors has been proposed by Brune (1962) and Knopoff et al. (1964) 60 years ago, based on the relation involving the group velocity and low-loss media. For surface waves, the correction between the two quality factors can be significant. Brune (1962) found values from 1.2 to 1.3 for Rayleigh and Love waves. In the same manner, ultrasonic and resonant-bar measurements to estimate attenuation in the laboratory (see Gurevich and Carcione 2022, App. B1 and B2, respectively) yield different values. In the first case, the method uses a wave that travels through the sample in only one direction at a time; interference effects then do not occur. In the second case, a sample is made to oscillate in one of its eigenmodes so that a standing wave is formed. Such standing waves are created by the interference of propagating waves moving in opposite directions as a result of successive reflections at the ends of the sample. For instance, quasi-static numerical experiments (Santos et al. 2014) yield a temporal Q.

Constant Q with frequency is a concept that has been used in many studies so far to correctly model the amplitude and phase of seismic signals and to reduce the number of parameters of the inversion process. Spatial fractional derivatives have been used in the geophysical literature to model constant Q since Carcione (2010) introduced the fractional Fourier pseudospectral method. The reason for this is that the spatial fractional derivative does not require additional variables and computer storage like the standard linear solid rheology or the temporal fractional derivative (e.g., Blanch et al. 1995; Mainardi 2022; Carcione 2022, Section 3.11.1). The fractional pseudospectral method in Carcione (2010) computes non-integer order Laplacian derivatives. This approach implies anelastic attenuation and velocity dis-

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persion when implemented in wave equations. Using this method, Carcione and coworkers simulated the propagation of constant-Q waves in a series of publications (e.g., Zhu and Carcione 2013; Qiao et al. 2019).

The analysis of standing and propagating waves presented here is based on a theoretical analysis using an energy balance given in Carcione (2022, section 2.3). If ω_I and α are the temporal and spatial attenuation factors, $\exp(-\omega_I t)$ and $\exp(-\alpha x)$ are the standing and propagating decays, respectively, where t and x are the time and spatial variables, respectively. In what follows, we illustrate the actual relation between the temporal and spatial quality factor using stress–strain relations based on spatial fractional derivatives, which has a power-law form in the wavenumber and frequency domains, respectively, and a constant quality factor. The theory is not restricted to low-loss media.

Seismic attenuation based on spatial fractional derivatives

Let us define the viscoacoustic stress–strain relation in 3D space,

$$\sigma = M_0 k_0^{-2\gamma} \nabla^{2\gamma} \epsilon \tag{1}$$

(Carcione 2010; Zhu and Carcione 2013), where σ is the stress, M_0 is a bulk modulus, ϵ is the trace of the strain tensor ($\epsilon = \partial_i u_i$), u_i are the displacements, γ is the loss parameter, and k_0 is a reference scaling wavenumber. The case $\gamma = 0$ gives the acoustic stress–strain relation. The operator $\nabla = \sum_i \hat{x}_i \partial_i, \; i = 1, 2, 3$ is the spatial differentiation vector, such that $\nabla^2 = \Delta$, the Laplacian. The Einstein implicit summation is assumed. Modeling the seismic attenuation with equation (1) (especially a constant Q) is very efficient because, unlike other approaches, it does not require memory variables or additional spatial derivatives (Carcione 2022, Section 3.11.2).

The equations of momentum conservation are

$$\partial_i \sigma = \rho \partial_t^2 u_i, \tag{2}$$

where ρ is the mass density.

We assume homogeneous viscoelastic waves (the wavenumber and attenuation vectors point in the same direction) (e.g., Carcione 2022; Section 3.3.1). A standing or stationary wave has the form

$$\exp[\mathrm{i}(\Omega t - \kappa_i x_i)], \quad \Omega = \omega + \mathrm{i}\omega_I, \tag{3}$$

where Ω is the complex frequency, ω is the angular frequency (a real quantity), the wavenumber components κ_i are real, and $\mathrm{i} = \sqrt{-1}$. The wavenumber is $\kappa = \sqrt{\sum_i \kappa_i^2}$.

On the other hand, a propagating wave is of the form

$$\exp[\mathrm{i}(\omega t - k_i x_i)], \quad k = \sqrt{\sum_i k_i^2} = \kappa - \mathrm{i}\alpha, \tag{4}$$

where k_i are the complex wavenumber components, and κ is the real wavenumber, which can be different from that of the standing wave.

Stationary waves: temporal quality factor

In this case, the wave remains in place and decays with time, describing stationary-wave loss. It has a damping factor of the form $\exp(-\omega_I t)$. A Fourier transform of (1) to the (real) wavenumber domain gives

$$\sigma = M\epsilon, \quad M = M_0 \left(-\mathrm{i}\frac{\kappa}{k_0}\right)^{2\gamma} = M_0 \left(\mathrm{i}\frac{\kappa}{k_0}\right)^{2\gamma}, \tag{5}$$

since $\nabla \equiv -\mathrm{i} \sum_i \hat{x}_i \kappa_i$ or $\nabla^2 \equiv -\kappa^2$, where M is the complex modulus, and we have used the property $(-1)^2 = 1$. In fact, the problem becomes one-dimensional, since there is a single wave mode and isotropy. Therefore, we can apply the theory presented in Carcione (2022, Section 2.3).

The complex velocity is

$$v_c = \frac{\Omega}{\kappa} = \sqrt{\frac{M}{\rho}} = \sqrt{\frac{M_0}{\rho}} \left(\mathrm{i}\frac{\kappa}{k_0}\right)^{\gamma}, \tag{6}$$

and the phase velocity is

$$v_p = \frac{\mathrm{Re}(\Omega)}{\kappa} = \mathrm{Re}(v_c) = v_0 \left(\frac{\kappa}{k_0}\right)^{\gamma},$$
$$v_0 = \sqrt{\frac{M_0}{\rho}} \cos\left(\frac{\pi\gamma}{2}\right). \tag{7}$$

The temporal quality factor is

$$Q_T = \frac{\mathrm{Re}(M)}{\mathrm{Im}(M)} = \cot \pi\gamma, \tag{8}$$

where Re and Im denote real and imaginary parts, respectively.

Then,

$$\gamma = \frac{1}{\pi}\arctan\, Q_T^{-1}. \tag{9}$$

Since $\omega = \kappa v_p$, we have

$$\kappa = \left(\frac{\omega k_0^{\gamma}}{v_0}\right)^{1/(1+\gamma)}. \tag{10}$$

Defining $\omega_0 = v_0 k_0$ and using (7), we obtain

$$v_p = v_0\left(\frac{\omega}{\omega_0}\right)^{\beta}, \quad \beta = \frac{\gamma}{1+\gamma}. \tag{11}$$

On the other hand, the attenuation factor is

$$\omega_I = \mathrm{Im}(\Omega) = \omega \tan\left(\frac{\pi\gamma}{2}\right), \tag{12}$$

linear with frequency, as expected for a constant-Q medium.

Another physical quantity is the group velocity, valid for low-loss media (e.g., Carcione 2022), which can be obtained as

$$v_g = \frac{\partial\omega}{\partial\kappa} = \left(\frac{\partial\kappa}{\partial\omega}\right)^{-1} = v_0^{\beta/\gamma}\left(\frac{\omega}{k_0}\right)^{\beta} = v_p, \tag{13}$$

where we have used equation (10) and $\omega_0 = v_0 k_0$.

Traveling waves: spatial quality factor

In this case, $k = \kappa - \mathrm{i}\alpha$ and the frequency is real. The wave attenuates with distance with a damping factor $\exp(-\alpha x)$. From Eqs. (1) and (2), we obtain

$$\nabla^2\sigma = \rho\partial_t^2\epsilon, \quad \text{or} \quad k^2 M = \rho\omega^2, \tag{14}$$

since $\nabla^2 \equiv -k^2$ and $\partial_t^2 \equiv -\omega^2$,

$$M = M_0\left(\frac{\mathrm{i}k}{k_0}\right)^{2\gamma} \quad \text{and} \quad v_c = \frac{\omega}{k} = \sqrt{\frac{M_0}{\rho}}\left(\frac{\mathrm{i}k}{k_0}\right)^{\gamma}. \tag{15}$$

From (14) and (15), we have

$$v_c = \frac{\omega}{k} = c v_1\left(\frac{\mathrm{i}\omega}{\omega_1}\right)^{\beta}, \quad v_1 = \frac{1}{c}\sqrt{\frac{M_0}{\rho}},$$
$$\omega_1 = c v_1 k_0, \quad c = \cos\left(\frac{\pi\beta}{2}\right). \tag{16}$$

The phase velocity is

$$v_p = \frac{\omega}{\kappa} = \left[\mathrm{Re}\left(\frac{1}{v_c}\right)\right]^{-1} = v_1\left(\frac{\omega}{\omega_1}\right)^{\beta}, \tag{17}$$

and the spatial quality factor is

$$Q = \frac{\mathrm{Re}(M)}{\mathrm{Im}(M)} = \frac{\mathrm{Re}(v_c^2)}{\mathrm{Im}(v_c^2)} = \cot\pi\beta. \tag{18}$$

Then,

$$\gamma = \frac{\arctan\, Q^{-1}}{\pi - \arctan\, Q^{-1}}. \tag{19}$$

The relation between the quality factors (8) and (18) is

$$\arctan\, Q_T^{-1} = \frac{\pi\, \arctan\, Q^{-1}}{\pi - \arctan\, Q^{-1}}. \tag{20}$$

On the other hand, the attenuation factor is

$$\alpha = -\mathrm{Im}(k) = -\omega\mathrm{Im}\left(\frac{1}{v_c}\right) = \frac{\omega}{Q}\left(\frac{\omega_1}{\omega}\right)^{\beta}, \tag{21}$$

almost linear with frequency in this case ($\omega^{1-\beta} \approx \omega$ if $\beta \ll 1$).

Moreover, the group velocity is

$$v_g = \frac{\partial\omega}{\partial\kappa} = \left(\frac{\partial\kappa}{\partial\omega}\right)^{-1} = \frac{v_p}{1-\beta} = (1+\gamma)v_p, \tag{22}$$

where we have used Eq. (17).

In both cases, stationary and traveling waves, the phase velocity depends on frequency as $\omega^{\gamma/(1+\gamma)} = \omega^{\beta}$.

Examples

Knopoff et al. relations

Knopoff et al. (1964) showed that the temporal and spatial quality factors are related as $Q_T = (v_p/v_g)Q$, where v_g and v_p are the spatial group and phase velocities, respectively. Moreover, they showed that $\omega_I = v_g\alpha$. In their calculations, they assumed low-loss media, isotropy and 1D propagation. They conclude that the Q_T's measured from observations of the Earth's free vibration modes must be modified when compared to the Q's at the same periods obtained from experiments with propagating waves.

For the present model, we have

$$Q_T = (1+\gamma)Q, \quad \text{and} \quad \omega_I = (1+\gamma)v_p\alpha. \tag{23}$$

Using Eqs. (8) and (18), and the property $\tan\theta = \theta$ (low-loss media) for $\theta \ll 1$, it is straightforward to show that the first relationship is verified. Similarly, the second relationship can be shown to be valid from Eqs. (12) and (21).

Waves and diffusion

Let us assume M_0 = 10 GPa, ρ = 2000 kg/m^3, such that v_0 = 2222 m/s, k_0 = 0.1/m and two cases: γ = 0.07 (wavelike) and γ = 0.4 (diffusion). In the first case, we have quality factors Q_T = 4.5 and Q = 4.8, typical of ocean-bottom unconsolidated sediments or mud (Hamilton 1972). In the diffusion case, Q_T = 0.3 and Q = 0.8. Graphical results are presented in Figs. 1

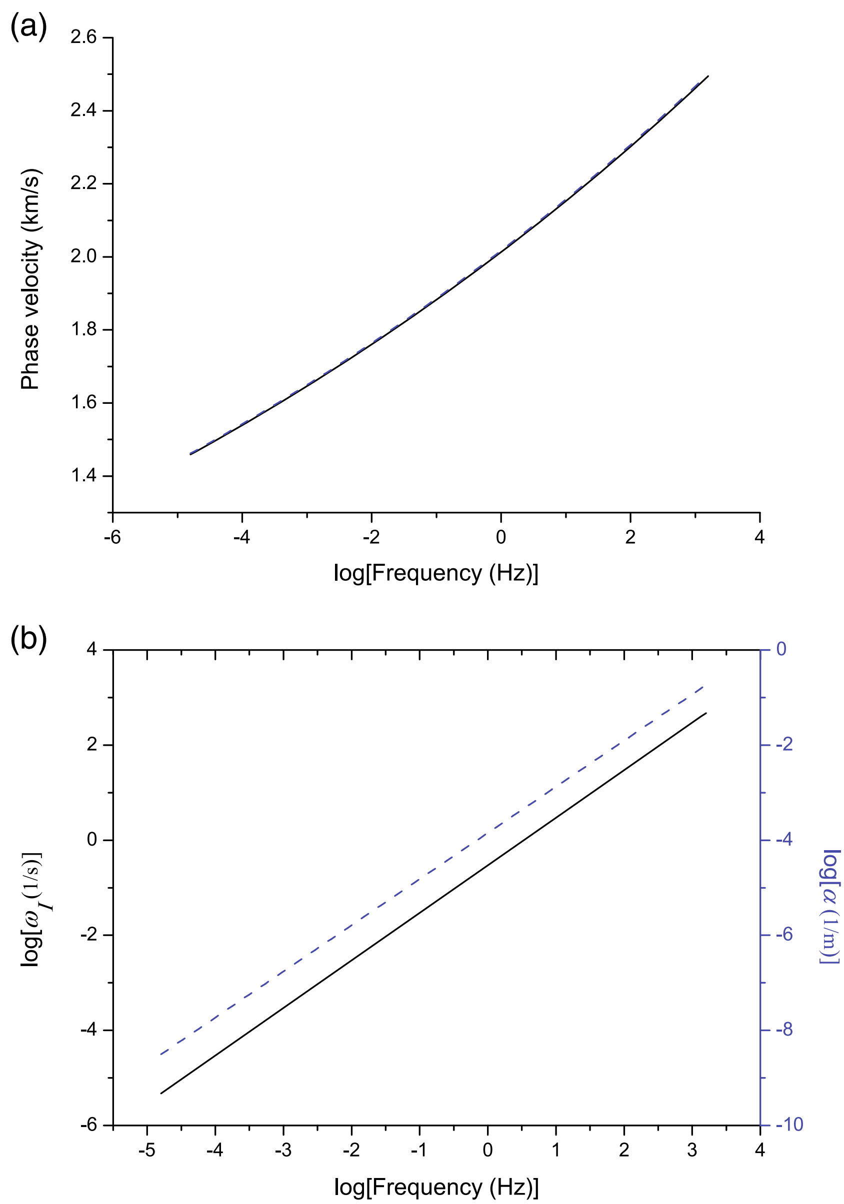

Fig. 1 Phase velocity (**a**) and attenuation factor (**b**) as a function of frequency for the wavelike case. The solid and dashed lines refer to standing waves and propagating modes, respectively. (Q_T = 4.5 and Q = 4.8 are constant)

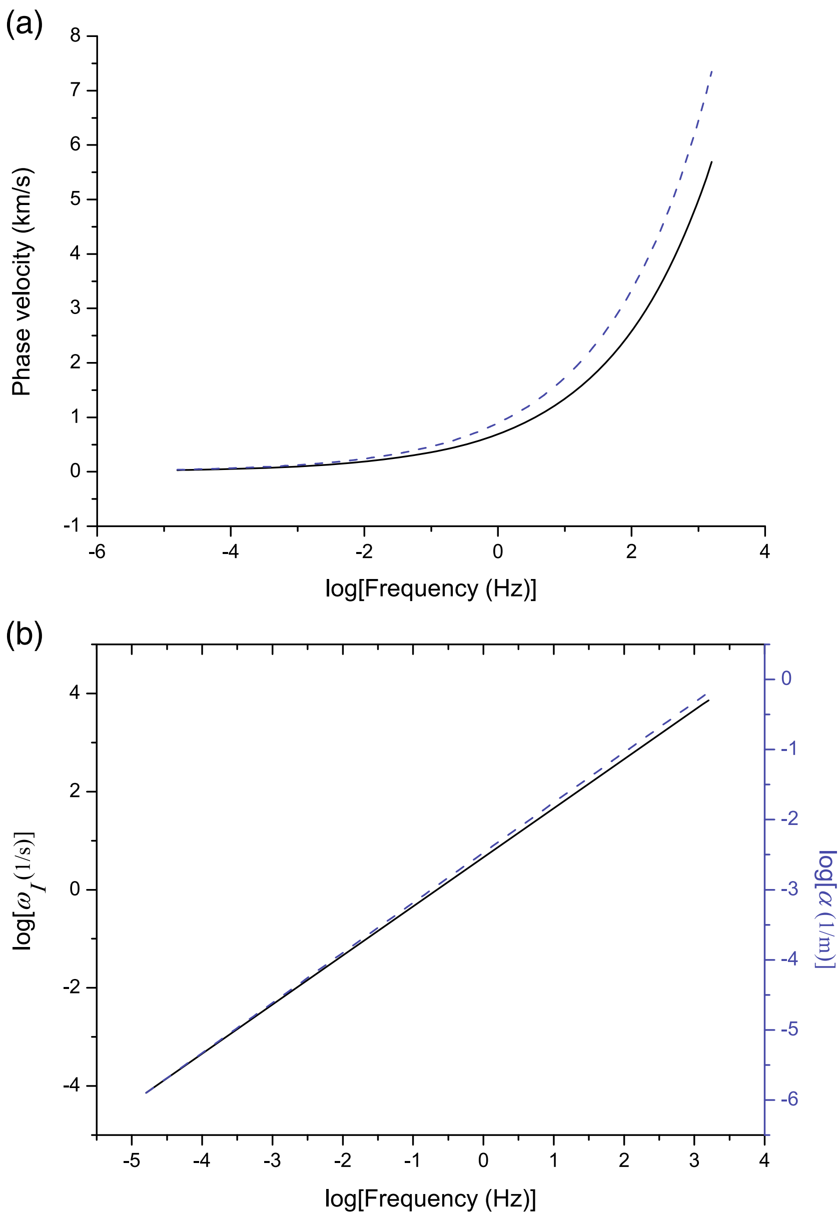

Fig. 2 Phase velocity (**a**) and attenuation factor (**b**) as a function of frequency for the diffusion case. The solid and dashed lines refer to standing waves and propagating modes, respectively. (Q_T = 0.3 and Q = 0.8 are constant)

and 2 for the wavelike and diffusion cases, respectively. The plots show the phase velocities (a) and attenuation factors (b) as a function of frequency for the temporal and spatial cases (dashed-blue and black-solid lines, respectively). In the case of diffusion, higher velocity and stronger attenuation can be observed.

Conclusions

The relationship between the temporal and spatial Qs has been obtained for a power-law stress–strain equation based on spatial fractional derivatives. The analysis provides explicit expressions for the quality factors, attenuation factors, and wave velocities as a function of frequency. In particular, the theory has been applied to compare the quality factors associated with free vibrations modes in unbounded media and propagating compressional waves. The relationships established in the 1960 s by Knopoff et al. are verified.

Data Availability The data are available from the corresponding author on request.

Declarations

Conflict of interest The authors declare no competing interests.

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